Notes on Algebraicity and Holonomy Criteria

Berkeley-Stanford Number Theory Seminar

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The purpose of this note is to write out some details about algebraicity and holonomy bounds due to Calegari-Dimitrov-Tang, following Section 2 of [7] and Section 2 of [6], for the Berkeley-Stanford Number Theory Seminar. All errors and pedantry are due to me.

1 History

We give some history regarding algebraicity/holonomy results, following [6, Section 2.1]. Since these are just rough notes, I'm not going to put all the citations in; the results in this overview can be found in [7] and [6], as well as [1, Chapter 5].

The first (nontrivial) theorem along these lines is the theorem of Borel:

Theorem 1.1 (Borel)**.** Let $f(x) \in \mathbf{Z}[\![x]\!]$ be a power series meromorphic on an open ball of radius strictly greater than 1. Then f is a rational function.

A brief word about the proof: we want to combine an upper and lower bound for the coefficients, using the convergence radius (for the upper bound) and the fact that f has integer coefficients (for the lower bound). This will be the idea we use in Section 2.

I think it's important to note the following S-integer generalization of Borel's theorem, used by Dwork to prove the rationality for zeta functions of varieties over finite fields:

Theorem 1.2 (Dwork)**.** Let K be a number field, S a finite set of non-archimedean places of K, and $f(x) \in \mathcal{O}_{K,S}[\![x]\!]$ a power series. If:

1. For all the $r+s$ infinite places of K, f determines a power series meromorphic on the open ball $D(0,R_i) \subseteq \mathbf{C}$, for each $i = 1,\dots,r+s$.
2. For each $\mathfrak{p} \in S$, f determines a power series meromorphic on the open ball $D(0,R_{\mathfrak{p}}) \subseteq \mathbf{C}_{\mathfrak{p}}$. Here the $\mathfrak{p}$-adic metric on $\mathbf{C}_{\mathfrak{p}}$ is uniquely induced from the normalized $\mathfrak{p}$-adic metric on K: a uniformizer for $\mathfrak{p}$ has absolute value $1/N(\mathfrak{p})$.

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3. $R \coloneqq \prod_{i=1}^{r+s} R_i^{a_i} \cdot \prod_{\mathfrak{p}\in S} R_{\mathfrak{p}} > 1$, where $a_i = 1$ if the ith infinite place is real, and $a_i = 2$ if it is complex.

Then f is a rational function.

Of course, Borel's theorem is the case $K = \mathbf{Q}, S = \emptyset$.

We now want to look at cases where we allow domains other than open disks.

Definition 1.3. Let $\Omega \subsetneq \mathbf{C}$ be a simply connected domain containing 0, so there is a biholomorphism $\varphi : \mathbf{D} \to \Omega$ with $\varphi(0) = 0$. By Schwarz Lemma, φ is unique up to rotations of $\mathbf{D}$, so $|\varphi'(0)|$ only depends on Ω. We call this the *conformal radius* $\rho(\Omega, 0)$ of Ω at 0.

Theorem 1.4 (Pólya)**.** *Let $f(x) \in \mathbf{Z}[\![x]\!]$ be a power series that has a meromorphic continuation to some simply connected domain Ω of conformal radius $\rho(\Omega, 0) > 1$. Then f is a rational function.*

This generalizes Borel's theorem, since the conformal radius of $D(0, r)$ is obviously r. Note that [1, Theorem 5.4.6] gives a p-adic generalization of Pólya's theorem due to Bertrandias, but we won't mention it here.

We may now want to generalize to the case where φ is no longer biholomorphic but still exhibits a large "conformal radius" $|\varphi'(0)|$. Here, the $f(x) \in \mathbf{Z}[\![x]\!]$ in question may no longer be rational, but we can find a substitute in that it is *algebraic*, i.e. it satisfies a polynomial equation over $\mathbf{Q}(x)$. This is done by [2, Chapter VIII] (and extended by [7, Theorem 2.0.1]), using a dimension bound that is modified in the below Theorem 2.1 (in particular take the b_i to be 0 in Theorem 2.1, so that the coefficients are integral).

Example 1.5 ([6], Remark 2.1.2)**.** Consider $f(x) = 1/\sqrt{1-4x}$, which has (binomial) expansion $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$ at 0. This is clearly an algebraic function but not rational. On the domain $\Omega = \mathbf{C} - [1/4, \infty)$ this is well-defined. The map $\psi(z) = -z/(z-1)^2$ is a biholomorphism $\mathbf{D} \to \Omega$ sending 0 to 0, so we calculate the conformal radius of Ω to be $|\psi'(0)| = 1$. So we barely miss out on the hypotheses for Pólya's theorem (so it is in some sense "sharp"). On the other hand, if λ is the modular lambda function in terms of $q \in \mathbf{D}$ (extended from its fundamental domains to the boundaries and cusp via $\varphi(0) = 0$), then $\varphi(z) \coloneqq \alpha\lambda(z)$ has $\varphi(0) = 0$, $|\varphi'(0)| = 16\alpha$, and φ does not have any other preimages of 0 or α. With $\alpha = 1/4$, since f can be analytically continued along any path from 0 missing both 0 and α, we conclude that the pullback $f \circ \varphi$ is single-valued and holomorphic on $\mathbf{D}$, since φ is a *covering map* of $\mathbf{C} - \{0, \alpha\}$ and so we can uniquely lift such paths to domain of φ and avoid monodromy issues. We thus apply the algebraicity criterion to f and φ as above with $\alpha = 1/4$, so φ has conformal radius $|\varphi'(0)| = 4 > 1$, and hence f is algebraic.

Further algebraicity criteria can be found in the work of André [2], generalizing the algebraicity criterion to power series of multiple variables, and Bost–Chambert-Loir [4], bringing in the techniques of Arakelov theory to generalize to the case of formal functions on an algebraic curve (in the above discussion, this curve was just $\mathbf{P}^1$). We don't discuss this.

For [6], since we are now introducing controlled denominators into our coefficients, algebraicity is not good enough, because powers of such series will not exhibit the same controlled denominators (consider for instance $\log(1-x) = \sum_{n=1}^{\infty} -x^n/n$, which has controlled denominators in the below sense but is not algebraic). But derivatives of such series do exhibit the same controlled denominators, so that is why [6] looks at holonomy instead as a substitute.

2 The main holonomy (dimension) bound

Basically, we follow Sections 2.1, 2.2, and 2.4 of [7] to prove the following result:

Theorem 2.1. Let $\varphi : \overline{\mathbf{D}} \to \mathbf{C}$ be a holomorphic function on (an open neighborhood of) the closed unit disk such that $\varphi(0) = 0$. Given $b_1, \ldots, b_r \in \mathbf{Q}_{\geq 0}$ with sum b, consider power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{[1, \ldots, b_1 n] \cdots [1, \ldots, b_r n]}, \tag{2.1}$$

where each $a_n \in \mathbf{Z}$ and $[1, \ldots, j]$ denotes $\operatorname{lcm}(1, 2, \ldots, j)$ for a positive integer j, such that $f(\varphi(z))$ is the germ of a *meromorphic* function on $\overline{\mathbf{D}}$. Let $\mathcal{H}(\varphi; b_1, \ldots, b_r)$ be the $\mathbf{Q}(x)$-linear span of all such formal functions. Then, under the assumption that $|\varphi'(0)| > e^b$, we have

$$\dim_{\mathbf{Q}(x)} \mathcal{H}(\varphi; b_1, \ldots, b_r) \leq e \cdot \frac{\int_{\mathbf{T}} \log^+|\varphi| \mu_{\mathrm{Haar}}}{\log|\varphi'(0)| - b}. \tag{2.2}$$

Remark 2.2. When all the b_i are equal, we recover Equation (2.2.3) of [6].

Remark 2.3. We want holomorphicity of φ on $\overline{\mathbf{D}}$, not $\mathbf{D}$, so that we can integrate over the unit circle to get the upper bounds in the proof (see below). But due to a limiting argument (i.e. shrinking $\mathbf{D}$ very slightly so that φ is holomorphic on a closed disk, but still such that the conformal radius is strictly greater than e^b), this difference does not affect the condition on the conformal radius $|\varphi'(0)|$ in the theorem.

Corollary 2.4 ([6], Corollary 2.6.1)**.** Let f be a formal function as in Theorem 2.1. Under the same assumption that there exists a holomorphic function $\varphi : \overline{\mathbf{D}} \to \mathbf{C}$ sending 0 to 0 such that $f(\varphi(z))$ is the germ of a meromorphic function on $\overline{\mathbf{D}}$ and $|\varphi'(0)| > e^b$, then $f(x)$ is a *holonomic* function, i.e. there is a nonzero linear differential operator $\mathcal{L}$ with $\mathbf{Q}[x]$-coefficients such that $\mathcal{L}(f) = 0$.

We note that this *qualitative* result (as compared to the explicit dimension bound found in Theorem 2.1) can already be found in [2, Chapter VIII 1.6].

Proof. If f satisfies the given hypotheses, then so does zf', since $zf'(\varphi(z)) = z\frac{d}{dz}f(\varphi(z))/\varphi'(z)$ is holomorphic at 0 and meromorphic on $\overline{\mathbf{D}}$, and the multiplication by z (shifting up by one degree) ensures that our denominators are still of the form prescribed in (2.1). Then the derivatives of f all lie in $\mathcal{H}(\varphi; b_1, \ldots, b_r)$, which has finite $\mathbf{Q}(x)$-dimension by Theorem 2.1, so there is some finite $\mathbf{Q}(x)$-linear relation among them. Clearing denominators, we produce our desired linear differential operator $\mathcal{L}$ with $\mathbf{Q}[x]$-coefficients. □

Remark 2.5. The bound (2.1) is actually not strong enough to ultimately apply to the main goal of [6]. A refined bound is given in [6, Theorem 7.0.1], which will be the discussion of future talks; the main improvement is getting rid of the e-coefficient via a double integral in the numerator (already present in work of Bost-Charles [5]), and improving the $-b$ term in the denominator of (2.2). The improvement is barely enough to prove the main Theorem A of [6], but further improvements with a more comfortable margin can be found in Sections 6 and 7 of the same paper (Theorems 6.0.2, 7.1.6).

We now aim to prove Theorem 2.1. For notation, given a d-tuple of complex numbers $\mathbf{x} = (x_1, \ldots, x_d)$ and a d-tuple of integers $\mathbf{k} = (k_1, \ldots, k_d)$, we define $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_d^{k_d}$. If h is a function of a single variable, then $h(\mathbf{x}) = (h(x_1), \ldots, h(x_d))$.

We will need the following lemma. For notation, let $f_1, \ldots, f_m \in \mathbf{Q}[\![x]\!]$ be $\mathbf{Q}(x)$-linearly independent formal functions such that each f_i can be written in the form (2.1), and each pullback $f_i \circ \varphi$ is the germ of a meromorphic function on $\overline{\mathbf{D}}$. Because $\varphi(0) = 0$ and φ is not identically 0, the power series f_i must each have some positive radius of convergence, so pick $\rho > 0$ small enough such that each of the f_i are holomorphic on $\overline{D(0,\rho)}$.

Lemma 2.6 ([7], Lemma 2.1.2)**.** Let $d, \alpha \in \mathbf{N}$ and $\kappa \in (0, 1)$. Asymptotically in $\alpha \to \infty$ with d and κ held fixed, there is a nonzero d-variate formal function $F(\mathbf{x})$ of the form

$$F(\mathbf{x}) = \sum_{\mathbf{i} \in \{1,\ldots,m\}^d,\ \mathbf{k} \in \{0,\ldots,D-1\}^d} a_{\mathbf{i},\mathbf{k}} \mathbf{x}^{\mathbf{k}} \prod_{s=1}^{d} f_{i_s}(x_s)$$

vanishing to order at least α at $\mathbf{x} = 0$, such that

1. $$D \leq \frac{1}{(d!)^{1/d}} \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{1/d} \alpha + o(\alpha),$$

   and

2. all $a_{\mathbf{i},\mathbf{k}}$ are integers bounded in absolute value by $\exp(\kappa C \alpha + o(\alpha))$ for some $C \in \mathbf{R}$ depending only on ρ and the f_i's (in particular C does not depend on our parameters d, α, κ).

Proof. Via our choice of ρ, by Cauchy's integral formula the kth coefficient of f_i is at most $O(\rho^{-k})$ in absolute value for all k, albeit with the implicit constant depending on the f_i.

To construct F of the desired form vanishing to order at least α at $\mathbf{x} = 0$, we can solve a homogeneous linear system of $\sum_{i=0}^{\alpha-1} \binom{i+d-1}{d-1} = \binom{\alpha+d-1}{d} \sim_{\alpha\to\infty} \alpha^d/d!$ equations in the $(mD)^d$ coefficients $a_{\mathbf{i},\mathbf{k}}$. After clearing denominators (as the f_i need not have integral coefficients), Siegel's Lemma [3, Lemma 2.9.1] provides a nonzero integral solution to such a system, given that $(mD)^d > \binom{\alpha+d-1}{d}$, or asymptotically as $\alpha \to \infty$, that

$$D \sim \frac{1}{(d!)^{1/d}} \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{1/d} \alpha$$

for a parameter $\kappa \in (0, 1)$. Here we introduce κ for more control over the number of equations as compared to the number of indeterminates. Moreover, because we are only considering products of terms of the f_{i_s}'s of maximum degree α, and we cleared denominators by scaling the whole system by a height constant h (depending only the f_i's) such that $\log(h) = O(\alpha)$ (by the prime number theorem; see the below proof of the main theorem), the $a_{\mathbf{i},\mathbf{k}}$ can be taken to be bounded in logarithmic absolute value by

$$\frac{\alpha^d/d!}{(mD)^d - (\alpha^d/d!)} \left(\alpha O(\log \rho^{-1}) + \log(h) + d\log(mD)\right) \sim \kappa \left(\alpha O(\log \rho^{-1}) + O(\alpha) + o(\alpha)\right) = \kappa C\alpha + o(\alpha)$$

for some constant C only depending on ρ and the f_i's.

Finally, we need to show that F is not the 0 function. The f_i are $\mathbf{Q}(x)$-linearly independent, so the $f_{\mathbf{i}}(x_1, \ldots, x_d) := \prod_{s=1}^{d} f_{i_s}(x_s)$ (for $\mathbf{i} \in \{1, \ldots, m\}^d$) are $\mathbf{Q}(x_1, \ldots, x_d)$-linearly independent. Indeed, if we have a nontrivial linear relation of the $f_{\mathbf{i}}$ equaling 0, then we may choose some nonzero constant $c \in \mathbf{Q}$ (within the common convergence radius ρ of all the f_i) to obtain a nontrivial linear relation of the $\{\prod_{s=1}^{d-1} f_{i_s}(x_s)\}_{\mathbf{i}' \in \{1,\ldots,m\}^{d-1}}$ over $\mathbf{Q}(x_1, \ldots, x_{d-1})$. Repeating this process results in a nontrivial relation of the $f_i(x_1)$ over $\mathbf{Q}(x_1)$, a contradiction. So as not all the coefficients $a_{\mathbf{i},\mathbf{k}}$ are 0 in the definition of F, we conclude that F is nonzero by this $\mathbf{Q}(x)$-linear independence. □

Let's complete the proof of the main theorem. For this, we may choose a single nonzero holomorphic function h on $\overline{\mathbf{D}}$ such that $h \cdot f_i \circ \varphi$ is holomorphic on $\overline{\mathbf{D}}$ for all i, and scaled so that $h(0) = 1$.

Proof of Theorem 2.1. Let's try to spell out the idea first, following [7, Section 1.1]. The idea is to mimic the (very easy) proof that a power series with integer coefficients converging on an open disk of radius > 1 is a polynomial—we use the Cauchy integral formula to show the coefficients go to 0, but a nonzero integer has absolute value at least 1. We try to do the same here, using the auxiliary function F constructed in Lemma 2.6. The upper bound (called the "Cauchy bound" in [7]) comes from our bounds on the coefficients $a_{\mathbf{i},\mathbf{k}}$ in that lemma. The lower "Liouville" bound comes from our knowledge of the "shape" of the coefficients of the f_i's (2.1). Playing these off each other should get us the desired bound on m, since as we increase the vanishing order α, the $a_{\mathbf{i},\mathbf{k}}$ remain small in the logarithmic scale, so the lowest-order coefficient retains a Cauchy upper bound inversely in m, but it also cannot be *too small* due to the Liouville bound, so F could not have been constructed from "too many" f_i's.

Consider $G(\mathbf{z}) = G(z_1, \ldots, z_d) \coloneqq \prod_{s=1}^{d} h(z_s) F(\varphi(z_1), \ldots, \varphi(z_d))$ with F as in Lemma 2.6. This is holomorphic on the closed d-dimensional polydisk $\{\mathbf{z} : \max_{i=1}^{d} |z_i| \leq 1\}$, because all of the factors φ and $h \cdot f_i \circ \varphi$ are. Let $c\mathbf{z}^{\mathbf{n}}$ be the lexicographically lowest monomial in $G(\mathbf{z})$ (note that $G(\mathbf{z})$ is not identically 0 since F is not identically 0 and φ is locally a biholomorphism near 0). We have the following lemma:

Lemma 2.7 ([7], Lemma 2.4.2)**.** Let $G(\mathbf{z}) \in \mathbf{C}[\![\mathbf{z}]\!] - \{0\}$ be holomorphic on the closed d-dimensional unit polydisk. If $c\mathbf{z}^{\mathbf{n}}$ is the lexicographically minimal monomial of G, then

$$\log|c| \leq \int_{\mathbf{T}^d} \log|G| \mu_{\mathrm{Haar}}.$$

Sketch. The proof reduces to the 1-variable case by induction, integrating over one variable at a time. In that case, because $z^{-n}G(z)$ is holomorphic by assumption, $\log|z^{-n}G(z)|$ is subharmonic with value c at 0, so that $\log|c| \leq \int_{\mathbf{T}} \log|z^{-n}G(z)|\mu_{\mathrm{Haar}} = \int_{\mathbf{T}} \log|G(z)|\mu_{\mathrm{Haar}}$ since $|z| = 1$ on $\mathbf{T}$. □

Then with our specific G and $c\mathbf{z}^{\mathbf{n}}$, we have

$$\log|c| \leq \int_{\mathbf{T}^d} \log|G| \mu_{\mathrm{Haar}} = \int_{\mathbf{T}^d} \left(\sum_{s=1}^{d} \log|h(z_s)| \right) + \log|F(\varphi(z_1), \ldots, \varphi(z_d))| \mu_{\mathrm{Haar}}. \tag{2.3}$$

The integral of the first sum is simply some constant not depending on α, d, or κ, which is $o(\alpha)$.

Next,

$$\begin{aligned}
\log|F(\varphi(z_1),\dots,\varphi(z_d))| &\le \log\left(\sum_{\mathbf{i}\in\{1,\dots,m\}^d,\ \mathbf{k}\in\{0,\dots,D-1\}^d} |a_{\mathbf{i},\mathbf{k}}\varphi(\mathbf{z})^{\mathbf{k}}\prod_{s=1}^d f_{i_s}(\varphi(z_s))|\right)\\
&\le \log\left(\sum_{\mathbf{i}\in\{1,\dots,m\}^d,\ \mathbf{k}\in\{0,\dots,D-1\}^d} \exp(\kappa C\alpha+o(\alpha))\prod_{i=1}^d \max(1,|\varphi(z_i)|)^{k_i}E\right)\\
&\le \log\left((mD)^d\exp(\kappa C\alpha+o(\alpha))\prod_{i=1}^d \max(1,|\varphi(z_i)|)^{D}E\right)\\
&\le d\log(mD)+\kappa C\alpha+o(\alpha)+D\sum_{i=1}^d \log^+|\varphi(z_i)|+\log(E)\\
&= \kappa C\alpha+o(\alpha)+D\sum_{i=1}^d \log^+|\varphi(z_i)|,
\end{aligned}$$

where $E=\max_{\mathbf{i}\in\{1,\dots,m\}^d}\prod_{s=1}^d|f_{i_s}(\varphi(z_s))|$ is a positive constant not depending on α or D. For the last line, note that $D\le\frac{1}{(d!)^{1/d}}\frac{1}{m}\left(1+\frac{1}{\kappa}\right)^{1/d}\alpha+o(\alpha)$ implies $\log(mD)=o(\alpha)$. Hence from (2.3),

$$\log|c|\le\int_{\mathbf{T}^d} D\sum_{i=1}^d\log^+|\varphi(z_i)|\mu_{\text{Haar}}+\kappa C\alpha+o(\alpha)\le dD\int_{\mathbf{T}}\log^+|\varphi(z)|\mu_{\text{Haar}}+\kappa C\beta+o(\beta). \tag{2.4}$$

Above, $\beta:=|\mathbf{n}|=\sum_{i=1}^d n_i$, and F vanishes to order at least α at $\mathbf{x}=0$ by construction, so $\beta\ge\alpha$ (in particular any function that is $o(\alpha)$ is also $o(\beta)$).

Next, we know that if $c\mathbf{z}^{\mathbf{n}}$ is the lexicographically lowest monomial in $G(\mathbf{z})$, $\varphi(0)=0$ implies $\varphi(z)$ has power series $\varphi(z)=\varphi'(0)z+$higher order terms, and moreover $h(0)=1$, so that the lexicographically lowest monomial of $F(\mathbf{x})$ must also have exponent $\mathbf{n}=(n_1,\dots,n_d)$. Moreover each $f_{i_s}(x_s)$ has the form $\sum_{n=0}^\infty a_n\frac{x_s^n}{[1,\dots,b_1n]\cdots[1,\dots,b_rn]}$, so that the lexicographically lowest monomial in $F(\mathbf{x})$ has coefficient in

$$\left(\prod_{i=1}^d[1,\dots,b_1n_i]\cdots[1,\dots,b_rn_i]\right)^{-1}\mathbf{Z}$$

Since $G(\mathbf{z})=\prod_{s=1}^d h(z_s)F(\varphi(\mathbf{z}))$ and $h(0)=1$, it follows that

$$c\in\varphi'(0)^\beta/\left(\prod_{i=1}^d[1,\dots,b_1n_i]\cdots[1,\dots,b_rn_i]\right)\cdot\mathbf{Z},$$

so as c is nonzero,

$$\log|c| \geq \beta \log|\varphi'(0)| - \sum_{i=1}^{d} \sum_{j=1}^{r} \log[1, \ldots, b_j n_i].$$

Note that for an integer k, $[1, \ldots, k]$ differs from $[1, \ldots, k-1]$ if and only if k is the power of a prime, in which case $[1, \ldots, k]/[1, \ldots, k-1] = p$ if $k = p^l$. Therefore $\log[1, \ldots, k] = \sum_{i \leq k} \Lambda(k)$ with Λ the von Mangoldt function, using the prime number theorem $\lim_{k\to\infty} \left(\sum_{i\leq k} \Lambda(i)/k\right) = 1$, we get $\log[1, \ldots, b_j n_i] = b_j n_i + o(n_i) = b_j n_i + o(\beta)$ (since the n_i's are at most β by definition), so

$$\log|c| \geq \beta \log|\varphi'(0)| - \sum_{i=1}^{d} \sum_{j=1}^{r} (b_j n_i + o(\beta)) = \log|\varphi'(0)| - \sum_{i=1}^{d} (b n_i + o(\beta)) = \beta \log|\varphi'(0)| - b\beta + o(\beta) \tag{2.5}$$

since d is a fixed parameter (with respect to β).

Upon combining (2.4) and (2.5),

$$\log|\varphi'(0)| - b + o(\beta)/\beta \leq dD/\beta \int_{\mathbf{T}} \log^{+}|\varphi(z)|\mu_{\mathrm{Haar}} + \kappa C + o(\beta)/\beta.$$

Letting $\alpha \to \infty$ (so $\beta \to \infty$ as well, and crucially none of the constants appearing above depend on d or κ), and with the fact that

$$\frac{D}{\beta} \leq \frac{1}{(d!)^{1/d}} \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{1/d} \frac{\alpha}{\beta} + \frac{o(\alpha)}{\beta} \leq \frac{1}{(d!)^{1/d}} \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{1/d} + \frac{o(\alpha)}{\beta}$$

we conclude that

$$\log|\varphi'(0)| - b \leq \left(\frac{d}{(d!)^{1/d}} \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{1/d} \right) \int_{\mathbf{T}} \log^{+}|\varphi(z)|\mu_{\mathrm{Haar}} + \kappa C.$$

Then sending $d \to \infty$ and $\kappa \to 0$, we get

$$m \leq e \cdot \frac{\int_{\mathbf{T}} \log^{+}|\varphi(z)|\mu_{\mathrm{Haar}}}{\log|\varphi'(0)| - b}.$$

This proves Theorem 2.1, since we've shown that any set of $\mathbf{Q}(x)$-linearly independent formal functions in $\mathcal{H}(\varphi; b_1, \ldots, b_r)$ is at most size $e \cdot \frac{\int_{\mathbf{T}} \log^{+}|\varphi(z)|\mu_{\mathrm{Haar}}}{\log|\varphi'(0)| - b}$. □

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